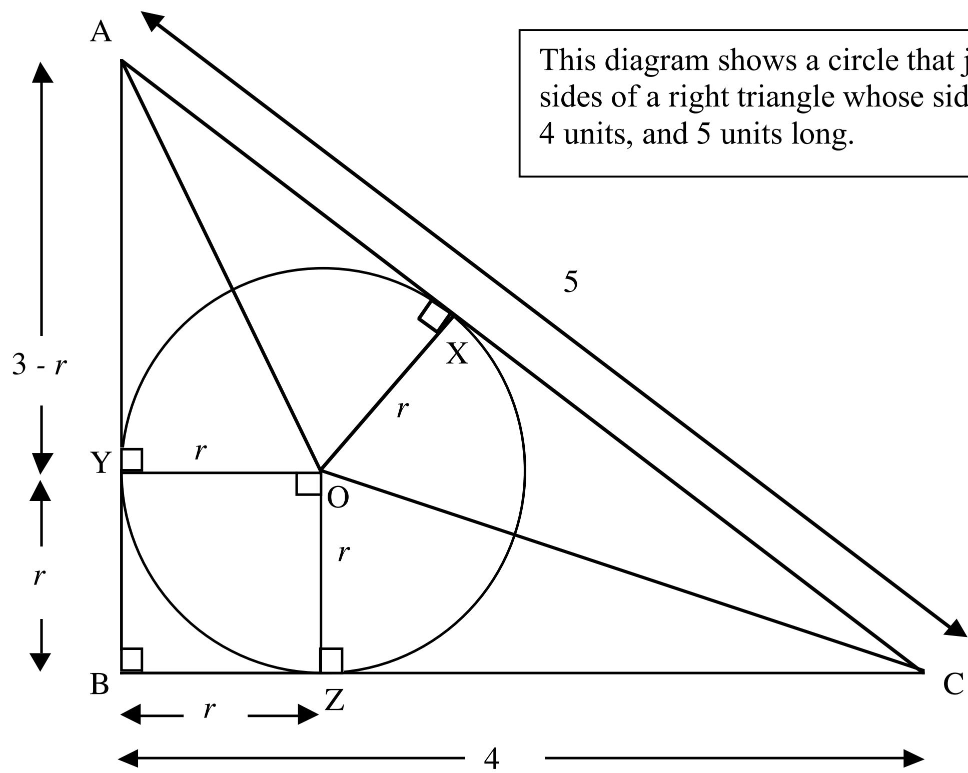
Circles in Triangles

This problem gives you the chance to:
• use algebra to explore a geometric situation



This diagram shows a circle that just touches the sides of a right triangle whose sides are 3 units, 4 units, and 5 units long.

1. Explain why triangles AOX and AOY are congruent.

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2. What can you say about the measures of the line segments CX and CZ?

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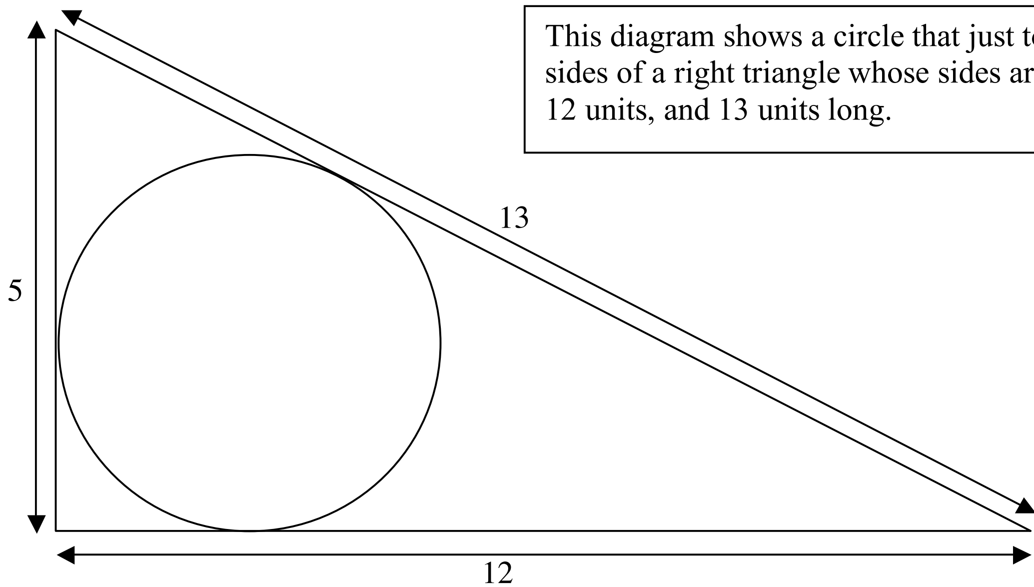
3. Find *r,* the radius of the circle. Explain your work clearly and show all your calculations.

4.



This diagram shows a circle that just touches the sides of a right triangle whose sides are 5 units, 12 units, and 13 units long.

Draw construction lines as in the previous task, and find the radius of the circle in this 5, 12, 13 right triangle. Explain your work and show your calculations.

9

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Circles in Triangles		Rubric
The core elements of performance required by this task are: • use algebra to explore a geometric situation Based on these, credit for specific aspects of performance should be assigned as follows	points	section points
1. Triangle AOY is congruent to triangle AOX (SAS) or (SSS) or (Hypotenuse, leg) *Partial credit* For a partially correct explanation	2 (1)	 2
2. States that: CZ = CX	1	1
3. Gives correct answer **r = 1** Shows correct work such as: CX = 4 – r = 3 = CZ CX = 5 - AX $5 = 3 - r + 4 - r$ or $r^2 + r(3- r) + r(4 - r) = 6$ *Partial credit* Shows partially correct work or uses guess and check. Alternatively, may use area reasoning	1 2 (1)	 3
4. Draws in construction lines and uses a similar method to Question #3, Gives correct answer **r = 2** Shows correct work such as: $13 = 5 - r + 12 - r$ or $r^2 + r(5- r) + r(12 - r) = 30$ Alternatively, may use area arguments.	1 1 1	 3
Total Points		**9**

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Circles in Triangles

Work the task and look at the rubric. What are the big mathematical ideas that students need to understand to work this task?

What are the rules or standards for making proofs in your classroom? What classroom experiences help students to grow in their understanding of the requirements or logic needed for a proof or convincing argument?

Look at student work for the proof that triangle AOY is congruent to AOX.

- How many of your students assumed that angle A was bisected?
- How many of your students tried to use SAS, but used the wrong sides?
- How many students made statements for why they were congruent, like SAS, SSS, or Hypotenuse Leg but failed to give any details to support their claim?
- How many students used inappropriate or unrelated mathematical ideas, such as making statements about alternate interior angles?

Some big names in mathematics education talk about quality feedback as one of the most important learning tools for students. How do you routinely give students feedback to help them improve their quality of explanations?

Now look at proofs or convincing arguments that received full marks. Which explanations give all the details and clarity that you value in a mathematical explanation? What are the characteristics that make these explanations different?

How do you communicate these ideas and values to students?

Do students get opportunities to read and critique the explanations or proofs of other students? How could this be incorporated into the classroom routine?

Look at part 3 of the task. Can you think of strategies that successful students might use to solve for the radius?

- How many of your students could find the radius using substitution?
- Using area of component parts?

A few students were able to correctly guess that r=1 for part 3 without giving any evidence to support that idea or by making incorrect assumptions. How many of your students were able to do this?

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Half the students in the sample did not attempt this part of the task? How many of your students fell into this category?

What might be some reasons for this? Do you think they don't understand the mathematical ideas needed for the solution? Do you think they don't have enough experience applying the geometrical ideas in non-routine, problem-solving situations?

Almost 2/3 of the students did not attempt part 4 of the task including drawing in the construction lines. How did your students do on this section of the task? What opportunities do students have that require them to draw in lines on a diagram to help them think about the problem and identify a solution path? What are some favorite problems that you might give students to help them develop this skill?

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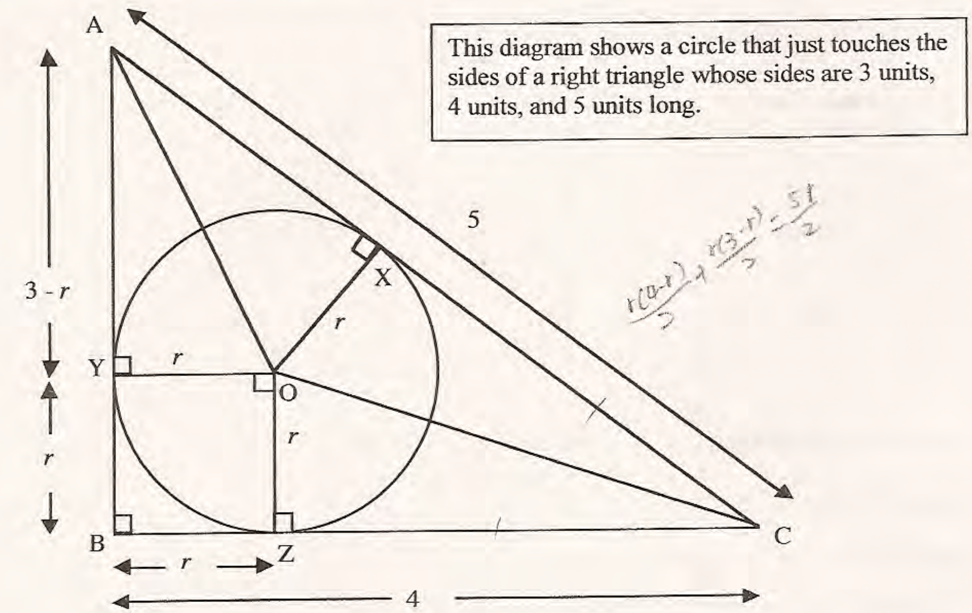
Looking at Student Work on Circles in Triangles

The significant story of this task is the quality of explanations and strategies students at this level are capable of using. Notice that the successful students mark up their diagrams to help them think (use the diagram as a tool). Successful students write complete explanations instead of just making short-hand statements about the situation. Students A and B use area strategies to solve for the radius in part 3 and 4. Student C uses a substitutions and a series of equations to solve for r.

Student A

Circles in Triangles

This problem gives you the chance to:
- use algebra to explore a geometric situation



1. Explain why triangles AOX and AOY are congruent.

$\triangle AOX \cong \triangle AOY$ by HL thm, because AO = AO by ref. prop., r = r by ref. prop., and $m\angle AYO = 90 = m\angle AXO$ by def. of ⊥ line. 2

2. What can you say about the measures of the line segments CX and CZ?

$\overline{CX} \cong \overline{CZ}$. $\triangle COX \cong \triangle COZ$ by HL thm, because CO = CO by ref. prop., r = r by ref. prop. and $m\angle CZO = 90 = m\angle CXO$ by def. of ⊥ line. Then $\overline{CX} \cong \overline{CZ}$ by CPCTC. 1

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Student A, part 2

3. Find *r*, the radius of the circle. Explain your work clearly and show all your calculations.

From questions 1 and 2, △AOX ≅ △AOY and △COX ≅ COZ, so △AOX + △COX = △AOY + △COZ ; therefore △AOC = △AOY + △COZ as $\frac{r(4-r)}{2} + \frac{r(3-r)}{2} = \frac{5r}{2}$. 1

Calculations: $\frac{r(4-r)}{2} + \frac{r(3-r)}{2} = \frac{5r}{2}$ 2

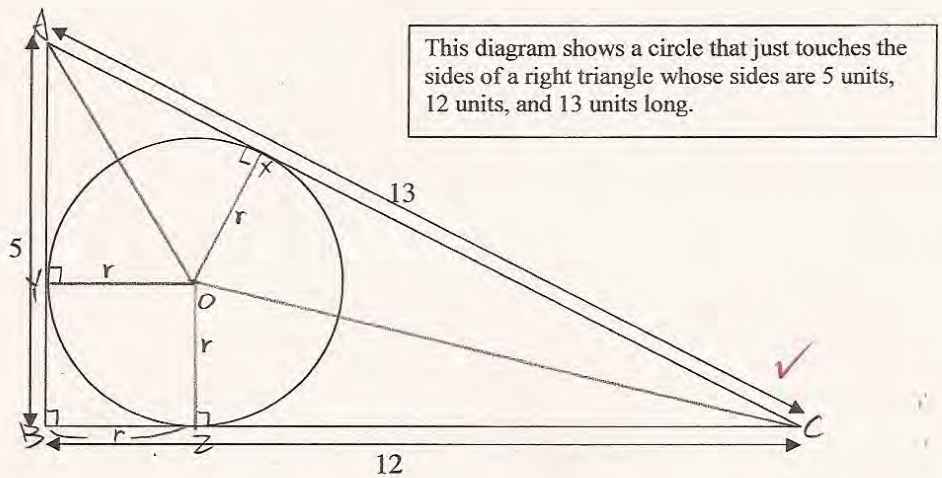
$r(4-r) + r(3-r) = 5r$ ✓

$4r - r^2 + 3r - r^2 = 5r$

$-2r^2 = -2r$

$r^2 = r$; $r = \pm 1$ Ans: $r = 1$ ✓

4.



Draw construction lines as in the previous task, and find the radius of the circle in this 5, 12, 13 right triangle. Explain your work and show your calculations.

△AOX ≅ △AOY by HL thm, △COX ≅ △COZ by HL thm. So △AOC = △AOY + △COZ 1

Calculations: $\frac{r(12-r)}{2} + \frac{r(5-r)}{2} = \frac{13r}{2}$ ✓ $\frac{r^2}{r} = 2$ 1

$12r - r^2 + 5r - r^2 = 13r$ $r = 2$ ✓ 1

$-2r^2 = -4r$

$r^2 = 2r$ Ans. $r = 2$

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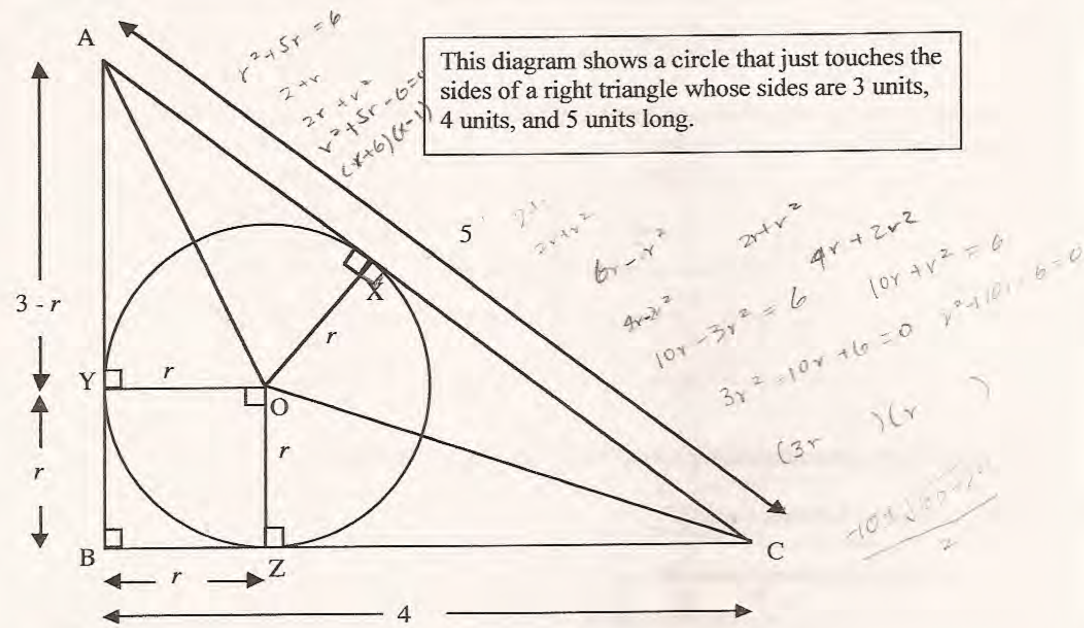
Student B

Circles in Triangles

This problem gives you the chance to:
• use algebra to explore a geometric situation



1. Explain why triangles AOX and AOY are congruent.

First $\overline{YO} \cong \overline{OX}$ because they are radii. Next, $\overline{AO} \cong \overline{AO}$ because of the reflexive property. Then, $\angle AYO \cong \angle AXO$ (90°). Therefore, $\triangle AOX \cong \angle AOY$ because of the "HL postulate." 2

2. What can you say about the measures of the line segments CX and CZ?

Their measures are the same: 2 + r because $\triangle COX \cong \triangle COZ$. 1

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Student B, part 2

3. Find r, the radius of the circle. Explain your work clearly and show all your calculations.

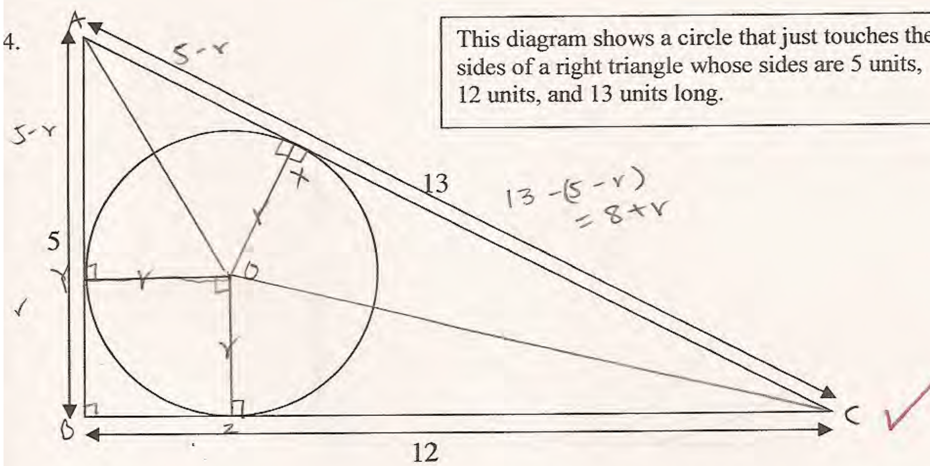
$\Delta AYO + \Delta AOX = (3-r)r = 3r - r^2$, $YOZB$ is r^2, $3r - r^2 + r^2 = 3r$,

$\Delta COX + \Delta COZ = (2+r)r = 2r + r^2$, $r^2 + 2r + 3r = \frac{1}{2}(3 \cdot 4)$, $r^2 + 5r = 6$ /

$r^2 + 5r - 6 = 0$, $(r+6)(r-1) = 0$ ✓ $r = -6, 1$, $r > 0$ because r is a 2

distance. $r = 1$. ✓

4.



This diagram shows a circle that just touches the sides of a right triangle whose sides are 5 units, 12 units, and 13 units long.

Draw construction lines as in the previous task, and find the radius of the circle in this 5, 12, 13 right triangle. Explain your work and show your calculations. /

$\Delta AYO + \Delta AOX = (5-r)r = 5r - r^2$, $YOZB$ is r^2, so $5r - r^2 + r^2$

$= 5r$, $\Delta COX + \Delta COZ = (8+r)r = 8r + r^2$, $r^2 + 8r + 5r = \frac{1}{2}(12 \cdot 5)$, /

$r^2 + 13r = 30$ $r^2 + 13r - 30 = 0$ $(r+15)(r-2)$, $r = -15, 2$, ✓ /

$r > 0$ because r is a distance. $r = 2$ ✓

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Student C

3. Find *r*, the radius of the circle. Explain your work clearly and show all your calculations.

$\overline{BZ} \cong \overline{BY} = r$

$\overline{AY} \cong \overline{AX} = x$

$\overline{CX} \cong \overline{CZ} = y$

$r + x = 3 \qquad r + x = 3$

$x + y = 5 \qquad -(y + x = 5)$

$y + r = 4 \qquad r - y = -2$

$r = y - 2$

$y + (y-2) = 4$

$2y = 6$

$y = 3$

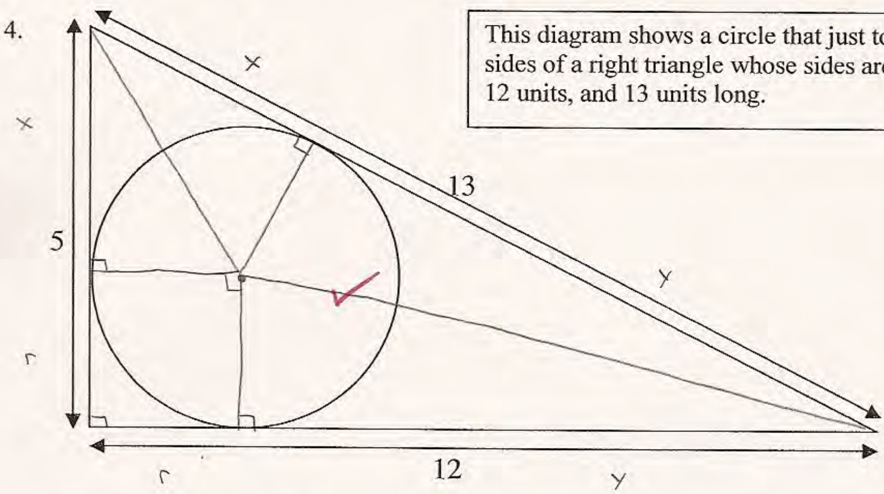
$y + r = 4$

$3 + r = 4$

$r = 1$

3

4.



This diagram shows a circle that just touches the sides of a right triangle whose sides are 5 units, 12 units, and 13 units long.

3

Draw construction lines as in the previous task, and find the radius of the circle in this 5, 12, 13 right triangle. Explain your work and show your calculations.

$r + x = 5 \longrightarrow r + x = 5$

$x + y = 13 \longrightarrow -(y + x = 13)$

$y + r = 12 \qquad r - y = -8$

$r = y - 8$

$y + (y-8) = 12$

$2y = 20$

$y = 10$

$y + r = 12$

$10 + r = 12$

$r = 2$

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Geometry Task 3 Sum and Product

Student Task	Use arithmetic and algebra to represent and analyze a mathematical situation and solve a quadratic equation by trial and improvement.
Core Idea 3 Algebraic Properties and Representations	**Represent and analyze mathematical situations and structures using algebraic symbols.** • Solve equations involving radicals and exponents in contextualized problems.

Mathematics in this task:
- Read and interpret a diagram to look for geometrical relationships
- Understand congruency and make convincing arguments about congruency from a diagram for sides, angles, line segments, and triangles
- Understand and use appropriately proofs for congruency, such as Side – Angle – Side, Side-Side-Side, and Hypotenuse- Leg
- Be able to solve a non-routine problems using substitution or comparing area to find a dimension in a complex diagram
- Being able to add lines to a diagram to help solve a problem

Based on teacher observations, this is what geometry students knew and were able to do:
- Recognize that two sides were equal in part 2 of the task
- State some of the supporting evidence for congruency in part 1, usually that the triangles shared a hypotenuse, that one side of each triangle was the radius of the circle and therefore equal, and that both triangles had right angles.
- Some students were able to guess the radius in part 3 or draw in the construction lines in part 4

Areas of difficulty for geometry students:
- Explaining how they know things are equal
- Making assumptions based on looking at relationships in a diagram, rather than proving the relationships
- Giving supporting evidence for their claims
- Unwillingness to attempt non-routine problems
- Understanding which construction lines are relevant to solving the problem or omitting details like making the lines from the center of the circle to the opposite sides right angles

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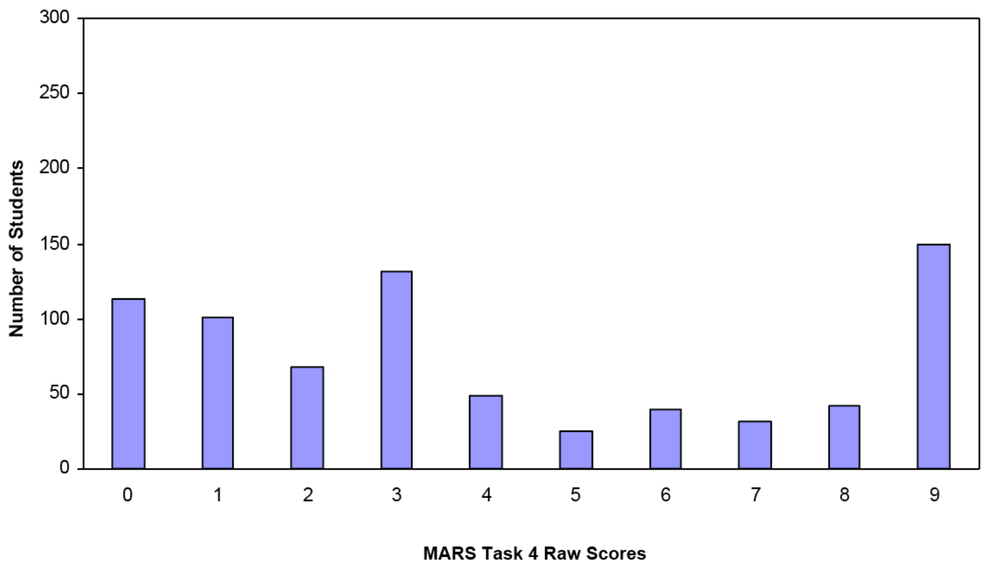
Task 4 - Circles in Triangles

Mean: 4.12 StdDev: 3.26

Table 53: Frequency Distribution of MARS Test Task 4, Course 2

Task 4 Scores	Student Count	% at or below	% at or above
0	113	15.1%	100.0%
1	101	28.5%	84.9%
2	68	37.6%	71.5%
3	131	55.1%	62.4%
4	49	61.6%	44.9%
5	25	64.9%	38.4%
6	40	70.3%	35.1%
7	32	74.5%	29.7%
8	42	80.1%	25.5%
9	149	100.0%	19.9%

Figure 62: Bar Graph of MARS Test Task 4 Raw Scores, Course 2



The maximum score available for this task is 9 points.
The minimum score for a level 3 response, meeting standards, is 4 points.

Many students, about 85%, could identify that line segments CZ and CX were equal. More than half the students, 63%, could explain with justi cation why two triangles were congruent and explain that segments CZ and CX were equal. Almost half, 45%, were also able to draw construction lines on a diagram. 20% of the students met all the demands of the task including finding the radius of circles inscribed in right triangles by using properties of congruence or area. 15% of the students scored no points on this task. 30% of the students with this score attempted the task.

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Circles in Triangles

Points	Understandings	Misunderstandings
0	30% of the students with this score attempted the problem.	Students could not make a correct statement about segments CX and CZ. Comments included "they're tangents, but not equal", "not congruent", "both intersect at C", and "they're flat".
1	Students knew that CX=CZ	Students could not give evidence to support why triangles were congruent in part 1. 12% assumed that the line through A was a bisector, with no supporting evidence. About 10% named a theorem with no supporting evidence.
3	Students could prove that the triangles were congruent and that CX=CZ.	More than half the students did not attempt part 3 of the task or did not attempt to draw construction lines in 4. Those who tried to draw construction lines made errors, usually they did not make the lines from the center to the opposite side form right angles or they left out the lines from the vertex to the center.
4	Students could solve part 1 and 2 and guess the size of the radius in 3 or draw in the construction lines in 4.	72% of the students did not try to solve for the radius in part 4.
7		Students could not solve for the radius in part 4, even though could solve a similar problem in part 3.
9	Students could read and interpret a diagram. Use geometric relationships to prove congruency of triangles in a complex diagram, and use substitution or area for the side measure in the diagram. Students at this level understood the importance of evidence to back up their claims and the logic needed to make a complete and convincing argument.	

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Implications for Instruction

For students to be successful on this task, they would need to have knowledge and utility with proofing congruent triangles. This might require more than a basic familiarity with two-column proofs and using fundamental theorems of congruency such as SSS, SAS, ASA, SAA, HL and LL. Students need to understand how valid geometric arguments are made. They also need to learn to apply this knowledge to non-routine situations. The problem may appear complex to the students because it involved more than one set of congruent triangles. In addition the inscribed circle may have distracted students from visualizing the sets of congruent triangles. Students need experiences with applying their knowledge to complex diagrams. Students must understand the importance in terms of flexibility having a right angle in a triangle. If the angle in the diagram was not a right angle then there would not have been enough information to justify that the triangles were congruent due to the fact that there is no SSA theorem. Providing opportunities for students to sort through when there is enough or not enough information is important to developing the reasoning and logic that underlies justification.

Another challenge to the task was taking the information in parts one and two and using that information to solve part three of the task. Parts one and two could have provided a scaffold for the students to prove that Triangle AYO was congruent to Triangle AXO and that Triangle CZO was congruent to Triangle CXO. Students also needed to use the knowledge that the corresponding parts of the triangles were congruent. The final step of logic was to use algebraic reasoning to determine a value for the radius. Three steps of reasoning often challenge students. They must have opportunities to reason over multi-step problems. Teachers should resist always breaking the reasoning sequence into parts for students, but rather teach them to think through the steps and develop a capacity to handle complex logic.

Part four of the problem involved generalizing from the initial problem and applying that generalization to a similar but different size triangle. Generalization and application are higher level thinking skills. Students need to be given opportunities to generalize and then apply the findings to a new situation. Often problems are solved for the sake of answers. In geometry, finding patterns, processes or formulas should be an initial step that then is utilized in new situations and/or proved to be a general tool for further exploration.

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Performance Assessment Task
Circles in Triangles Grade 10
This task challenges a student to use geometric properties of circles and triangles to prove that two triangles are congruent. A student must be able to use congruency and corresponding parts to reason about lengths of sides. A student must be able to construct lines to make sense of a diagram. A student needs to use geometric properties to find the radius of a circle inscribed in a right triangle.
Common Core State Standards Math - Content Standards
<u>**High School – Geometry - Congruence**</u> **Prove geometric theorems.** G-CO.9 Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.* G-CO.10 Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°, base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.* <u>**High School – Geometry - Circles**</u> **Understand and apply theorems about circles.** G-C.2 Identify and describe relationships among inscribed angles, radii and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribe angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.* G-C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.
Common Core State Standards Math – Standards of Mathematical Practice
MP.3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and – if there is a flaw in an argument – explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even through they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **MP.6 Attend to precision.** Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully

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formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.
Assessment Results
This task was developed by the Mathematics Assessment Resource Service and administered as part of a national, normed math assessment. For comparison purposes, teachers may be interested in the results of the national assessment, including the total points possible for the task, the number of core points, and the percent of students that scored at standard on the task. Related materials, including the scoring rubric, student work, and discussions of student understandings and misconceptions on the task, are included in the task packet.

Grade Level	Year	Total Points	Core Points	% At Standard
10	2007	9	4	45%

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